### Simple, prototype low pass filter

Consider averaging every two consecutive samples. In the time domain, the output is given as;



This equation produces a low pass filter. To see why, let's consider its response at very low and very high frequencies. For a very low frequency input, the signal hardly changes from sample to sample. In fact, if the input signal has frequency 0, *x*[*n*]= *A*cos(2**0*n*/*fs*), then it is constant. This is known as DC input, since direct current electrical signals have this quality. For DC input, the output of this filter is identical to the input.

Now suppose we have a very high frequency signal, at half the sampling frequency. So *x*[*n*]=*A*cos(2**( *fs*/2*)n*/*fs*)= *A*cos(*n*). Thus, the signal switches sign from sample to sample, and the output of this filter is zero.

We can express this in the *Z* domain as . Its transfer function, written in positive powers of *z*, is

,

and the square magnitude of this transfer function is

.

Hence this acts as a low pass filter, allowing low frequencies to pass through to the output, but removing high frequency content. You can easily see that for *f*=0, ***z*=1)=1 and for *f*= *fs*/2, *z*=-1)=0. But what happens halfway, at *f*= *fs*/4? Here, = 2*f*/*fs*= **/2 and we see that |***z=j*)|2= 1/2.

We call this the cut-off frequency of our low pass filter. Generally, the cut-off frequency is where the frequency response makes the transition between two values. There are various ways a cut-off frequency may be formally defined, but one of the most effective is that frequency at which the square magnitude is halfway between its low and high value (in this case, halfway between 0 and 1).

### Changing the gain at the cut-off frequency

Consider a transformation, given by

.

This has a few important properties. Every value inside the unit circle is mapped to a value inside the unit circle, those outside are mapped to values outside the unit circle, and those on the unit circle stay on the unit circle. So we can view this transfer function as preserving stability. Furthermore, it maps *z*=1 to 1 and *z*=-1 to -1.

Suppose that at some frequency *G* we have

.

Then from

.

We now use *F* to map this frequency G to /2, so that the square magnitude of *H*(*F*(*z*)) at /2 is the same as the square magnitude of *H*(*z*) at G. That is, we need the transformation

.

From and we have,

.

Thus, to change the gain at the cut-off frequency /2, we replace each first order section,  from our prototype filter with the following;

.

### Shifting the cut-off frequency

So far, our cut-off frequency has been set to **/2. We would like to be able to set the cut-off frequency to any value between 0 and **. This would allow us to change the location that provides the transition between where frequencies are passed and blocked.

Consider another transformation, given by

,

where as before,

.

This transfer function will map some new cut-off frequency *c* to the cut-off frequency of our prototype filter, **/2.

By putting into and solving for ,

.

Let us consider any first order section (the transfer function can be written as first order polynomials), . To change the cut-off frequency, we replace *H*(*z*) with,

,

where ** is defined as in .

### Creating a shelving filter

We can construct a low shelving filter by transforming our prototype filter, such that the square magnitude response is transformed from *H*2 to (*G*2-1)*H*2­+1. This transformation changes the extreme square magnitudes 0 and 1 of a low pass design to 1 and *G*2.

Recall that the poles will push up the magnitude response for nearby frequencies, and the zeros will pull it down. For the low shelving filter, we want to keep the poles on the imaginary axis, giving a sharp cut-off frequency. But now we shift the zeros so that, for each first order section *H*(*z*=1)=*g*=*G*1/*N* and *H*(*z*=-1)=1. That is, if the first order section of a prototype low pass filter is written as



Then the first order section of the low shelving filter becomes,



So .

### Inverting the magnitude response

For a high pass filter, we want |***z*=1)|2=0, |*z*=-1)|2=1 and |*z*=*c*)|2=G2. So we simply apply the transformation *FHP*(*z*)=-*z.*



### Simple low pass to band pass transformation

Now we want to create a band pass filter from our simple filter with centre frequency /2. We would like to transform the frequency range 0 to  to the frequency range – to If this transformation is applied to the input before a low pass filter is applied, then our low pass filter becomes a band pass filter. To do this, consider a transfer function *F*(*z*) with the following constraints;

.

*F* will move the lower and upper cut-off frequencies to ±/2, where our prototype low pass filter has its cut-off frequency, and it will move the centre frequency to 0, where our prototype low pass filter has gain equal to 1.

We can solve Eq. to arrive at

.

This transfer function has second order polynomials in the numerator and the denominator. So our first order section has now become a second order section.

## Popular IIR Filter Design

We have everything we need to construct many of the most popular audio filters. We first construct a prototype filter with prescribed gain at /2. Then we show how we can use the transformations just described to turn this prototype filter into the filters given.

### Low pass

As mentioned, a low pass filter has a transfer function *HLP* with magnitude 1 at frequency 0 and magnitude 0 at frequency **=** (*f*=*fs*/2). That is, the magnitude of the lowest frequencies are unaffected and the highest frequencies are eliminated. At some cut-off frequency *c*, it has magnitude *Gc*, and this represents the transition where frequencies below *c* are considered passed and above this are rejected.

In order to generate a high order low pass filter*, w*e first generate our high order prototype filter. Then by applying these transformations to each first order section in our high order prototype filter, we can generate a high order low pass filter.

We first use Eq. to transform each first order section in the *N*thorder prototype filter such that each first order section has magnitude *Gc*1/*N*, rather than (1/2)1/(2*N*), at **/2 (that is, the whole filter has square magnitude *Gc* 2, rather than 1/2 at **/2). Then we apply Eq. to shift the cut-off frequency so that each first order section has magnitude *Gc*1/*N* at *c*, rather than at /2*.*

Consider a simple first order case where we define the gain at the cut-off frequency such that the square magnitude is the average of the two extremes, *Gc* 2=(02+12)/2=1/2.

We start with

.

At **/2, this filter has square magnitude 1/2. That is, this definition of gain at the cut-off is the same definition used in the prototype filter. So there is no need to change the gain at the cut-off frequency.

Now we shift the cut-off frequency from **/2 to *c*,

.

This simplifies to,

.

### Low shelf

A low shelving filter has a transfer function *HLS* with magnitude *G* at frequency **0 (representing the low shelf), magnitude 1 at frequency **=**, and magnitude *Gc* at some cut-off frequency *c*.

As before, we start with our prototype filter. We will change the gain at the cut-off frequency to some value *g*. We will transform the range of square magnitudes of the prototype filter to the range of square magnitudes of the shelving filter. That is, we want the extreme square magnitudes, 0 and 1, of the low pass filter, 0 and 1, to map to the extreme square magnitudes of the low pass filter, 1 and *G* 2, with *g*2 mapping to *Gc* 2. Thus,

.

So for each first order section of an *N*thorder prototype filter,we apply Eq. and to change the gain at the cut-off frequency to this new value *g*1/*N.* Then we transform each section to a shelving filter using Eq. , and then use Eq. to shift the cut-off frequency to *c*.

Consider again our example first order filter with the gain at the cut-off frequency defined such that the square magnitude is the average of the two extremes *Gc*2=(*G*2+12)/2. Then Eq. simplifies to

,

and as with the low pass filter, this choice of cut-off frequency is the same as that used in the prototype filter.

Now we create the shelf,

.

And finally, we shift the cut-off frequency,

,

which reduces to



### Gain at bandwidth

The peaking and notch filters have additional parameters that relate to bandwidth. We previously specified that the centre frequency is where the filter reaches its maximum or minimum value, and cut-off frequency is where the square magnitude is half way between its two extremes, but what about bandwidth? Well, the gain at bandwidth is defined similar to the gain at the cut-off frequency. That is, *GB* is defined using the arithmetic mean of the extremes of the square magnitude response, *GB*2=(1+*G*2)/2. So



This is thus the simplest definition, and one we will use in examples that show how to generate simple low order filters. Note, however, that we will introduce an alternate definition later when we discuss parametric equalizers.

### Band pass filters

For a band pass filter, *HBP*(=0)=0, *HBP*(=/2)=0, *HBP*(=*c*)=1, and for the upper and lower cut-off frequencies, |*HBP*(=*l*)|= *|HBP*(=*u*)|=*Gc*. Bandwidth is defined as *B=u*-*l*. So this filter is designed to only pass a range of frequencies around the cut-off frequency, and suppress all other content.

To design a high order band pass filter, we consider each first order section of our prototype filter. We change the gain at the cut-off frequency to *Gc* using Eq. , then shift the cut-off frequency to the bandwidth with Eq. , where *c* in Eq. is replaced with *B.* Finally we transform this to a band pass filter using Eq. .

For our first order filter with the gain at the cut-off frequencies defined such that the square magnitude is the average of the two extremes *Gc*2=(02+12)/2=1/2, again there is no need to change the gain at the cut-off frequency of the prototype low pass filter.

So we now shift the center frequency of the prototype filter to *B*.



Then transform to a band pass filter with bandwidth *B* and center frequency *c,* which gives

.

Note that these filters are twice the order of the previous examples since the band pass transformation doubles the order of the filter. Now there are zeros at both 1 and -1, but poles surround the center frequency, allowing for a sharp transition from attenuating to passing frequency content. If the bandwidth is increased, then the distance of the poles from the center frequency would also increase.

### Peaking and notch filters

For a peaking or notch filter, *HPN*(=0)=1, *HPN*(=/2)=1, *HPN*(=*c*)=*G*, and for the upper and lower cut-off frequencies, |*HPN*(=*l*)|= *|HPN*(=*u*)|=*Gc*. Bandwidth is defined as *B=u*-*l*. When *G* is greater than 1, this filter provides a boost around *c*and is known as a peakingfilter. When *G* is less than 1, this filter attenuates the frequency content near *c*and is known as a notchfilter. Clearly, if *G=*0, the filter completely removes frequency content near *c* and is a band stop filter.

To design a high order peaking or notch filter, we consider each first order section of our prototype filter. We use Eq. and to change the gain at the cut-off frequency. Next we transform this to a shelving filter using Eq. . Then we shift the cut-off frequency to the bandwidth with Eq. , where *c* in Eq. is replaced with *B.* Finally we transform this to a band pass filter using Eq. .

For our first order filter with the magnitude of the transfer function at bandwidth defined as we defined the magnitude at the cut-off frequency for the shelving filter, we can follow the same steps as were taken with the shelving filter to give Eq. , except now the center frequency is replaced by the bandwidth *B.*



Then transform to a band pass filter with bandwidth *B.*

.

To summarise, when we use the simple definition of bandwidth or cut-off frequency, the standard filters mentioned above all have relatively straightforward forms for first order designs (where bandwidth is not specified), and second order designs (for band pass, band stop and peaking/notch filters). These are given in Table 1.

Table . Transformations.

|  |  |
| --- | --- |
| Transformation | Transfer function |
| Change the gain at cut-off frequency |  |
| Shift the cut-off frequency |  |
| Invert the magnitude response |  |
| Lowpass to low shelf |  |
| Lowpass to bandpass |  |

Table . Transfer functions of common first and second order filters, and their equivalent forms based on simpler filters.

|  |  |  |
| --- | --- | --- |
|  | Transfer function | Steps |
| Prototype |  |  |
| 1st order low pass |  | Prototype 🡪 Change gain at cut-off 🡪 Shift cut-off frequency |
| 1st order low shelf |  | Prototype 🡪 Change gain at cut-off 🡪Create shelf🡪 Shift cut-off frequency |
| 2nd order band pass |  | Prototype 🡪 Change gain at cut-off 🡪 Shift cut-off frequency 🡪 Create bandpass |
| Peaking or notch filter |  | Prototype 🡪 Change gain at cut-off 🡪Create shelf🡪 Shift cut-off frequency🡪 Create bandpass |



Figure.1. Pole zero plot (top) and square magnitude response for a fourth order prototype low pass filter.



Figure 2. Shifting the centre frequency of a low pass filter.

Normalised frequency 

|H|2

0



/2

Normalised frequency 

|H|2

0



c

1

G2



1

1/2

/2



Figure 3. Shelving filter transformation.





Figure 4. Pole zero plot (top) and square magnitude response for a fourth order prototype low shelving filter (bottom). Compared to our prototype filter, it moves the zeros towards the pole positions on the imaginary axis.



Figure 5. Reversing the z domain to turn a low pass into a high pass filter.



Figure 6. Transforming a low pass filter into a band pass filter.



Figure.7. Pole zero plot for a 1st order (top left) and 4th order (top right) low pass filter with center frequency c=/4. On bottom, square magnitude response for the first order (solid line) and fourth order (dashdot line) filters.



Figure.8. Pole zero plot for a 1st order (top left) and 4th order (top right) high pass filter with center frequency c=/4. On bottom, square magnitude response for the first order (dashdot line) and fourth order (solid line) filters.



Figure.9. Pole zero plot for a 1st order (top left) and 4th order (top right) low shelving filter with center frequency c=/4 and gain at center frequency *G*=2. On bottom, square magnitude response for the first order (dashdot line) and fourth order (solid line) filters.



Figure.10. Pole zero plot for a 1st order (top left) and 4th order (top right) high shelving filter with c=/4 and *G*=2. On bottom, square magnitude response for the first order (dashdot line) and fourth order (solid line) filters.



Figure.11. Pole zero plot for a 2nd order (top left) and 8th order (top right) band pass filter with c=/4 and *B*=/8. On bottom, square magnitude response for the first order (dashdot line) and fourth order (solid line) filters.



Figure.12. Pole zero plot for a 2nd order (top left) and 8th order (top right) band stop filter with c=/4 and *B*=/8. On bottom, square magnitude response for the first order (dashdot line) and fourth order (solid line) filters.



Figure.13. Pole zero plot for a 2nd order (top left) and 8th order (top right) peaking filter with c=/4, *B*=/8 and *G*=2. On bottom, square magnitude response for the first order (dashdot line) and fourth order (solid line) filters.